Spreadsheet Modeling & Decision Analysis

A Practical Introduction to Management Science

4th edition

Cliff T. Ragsdale

Chapter 8

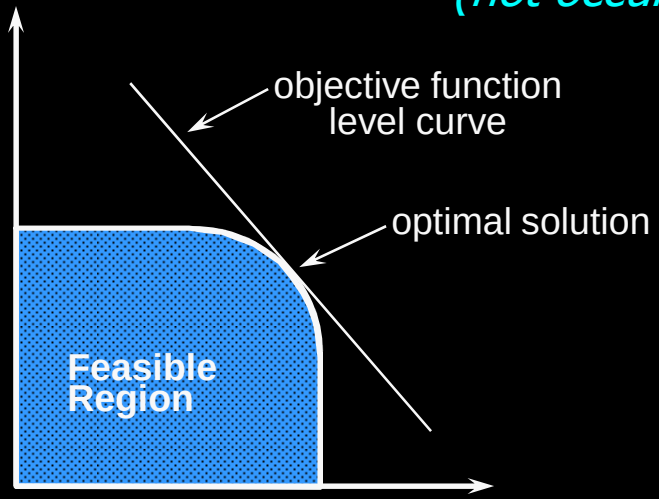
Nonlinear Programming & Evolutionary Optimization

Introduction to Nonlinear Programming (NLP)

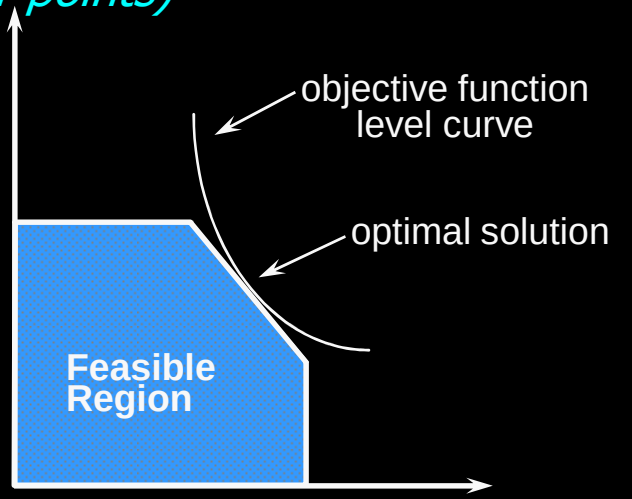
- An NLP problem has a nonlinear objective function and/or one or more nonlinear constraints.
- NLP problems are formulated and implemented in virtually the same way as linear problems.
- The mathematics involved in solving NLPs is quite different than for LPs.
- Solver tends to mask this different but it is important to understand the difficulties that may be encountered when solving NLPs.

Possible Optimal Solutions to NLPs

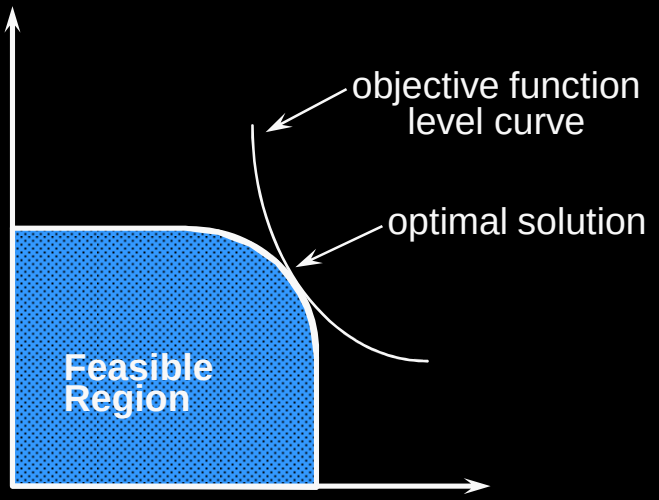
(not occurring at corner points)



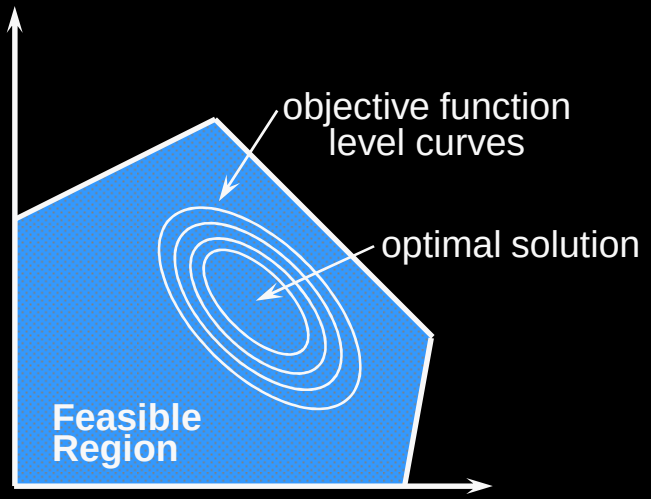
**linear objective,
nonlinear constraints**



**nonlinear objective,
linear constraints**



**nonlinear objective,
nonlinear constraints**

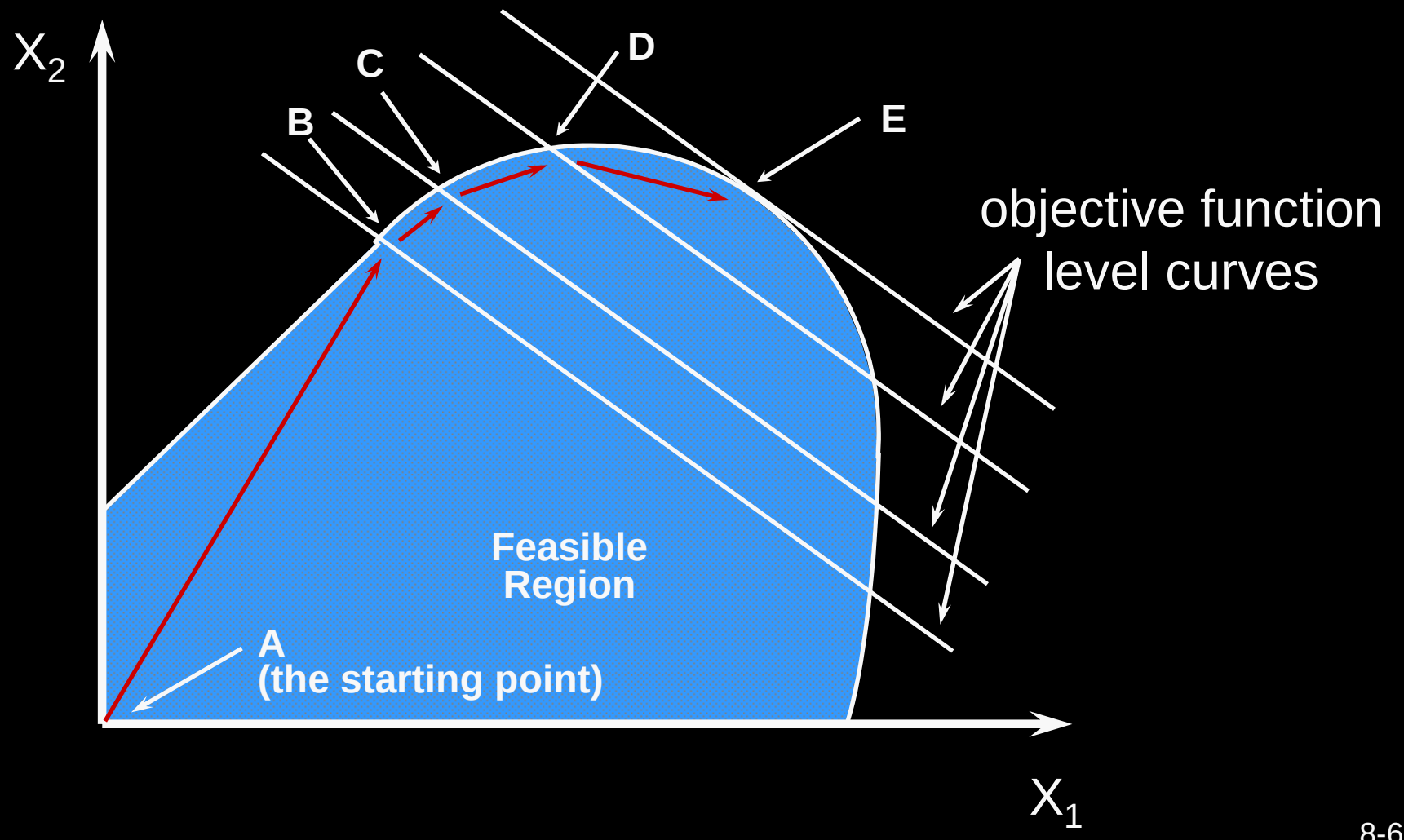


**nonlinear objective,
linear constraints**

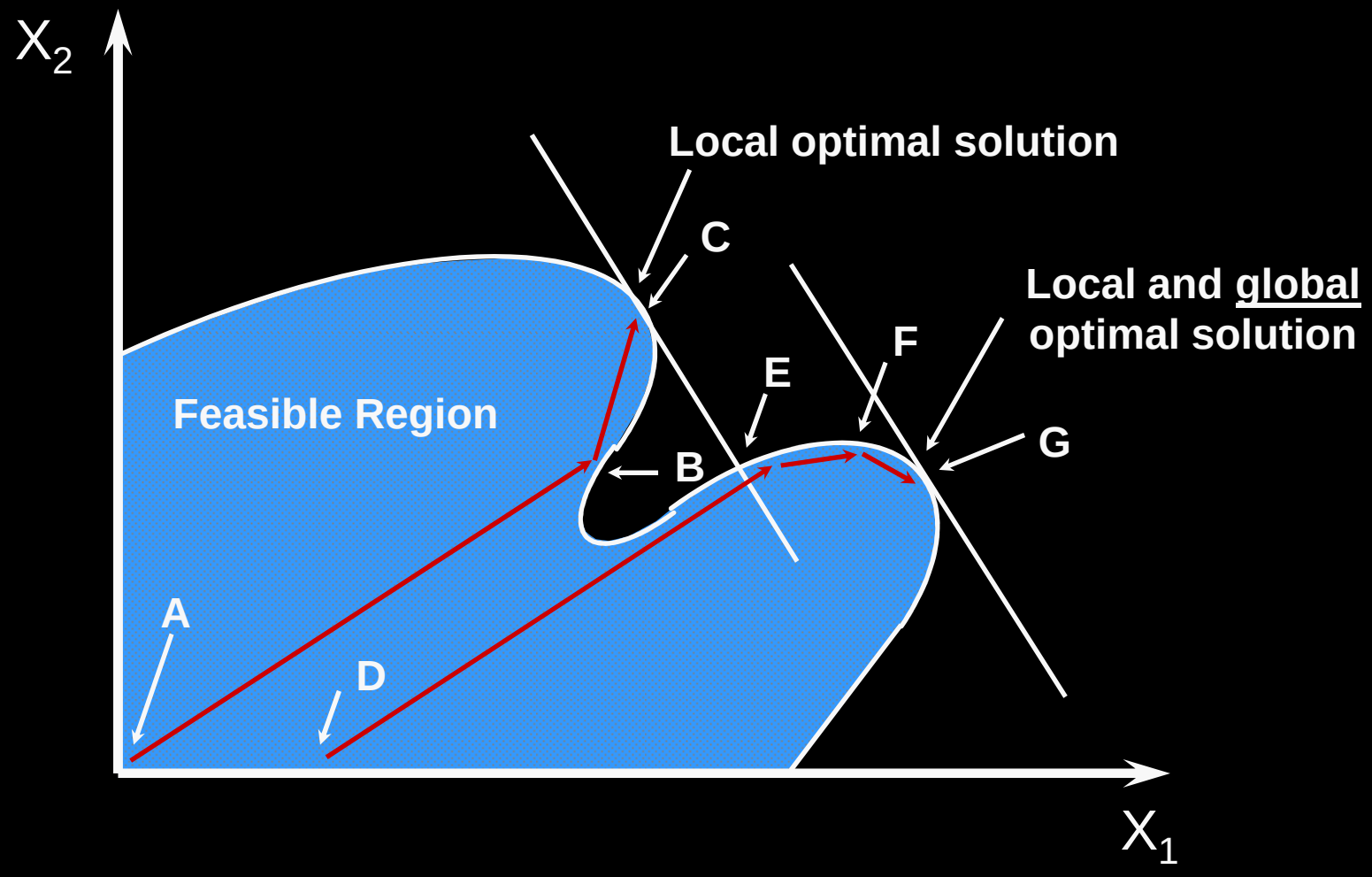
The GRG Algorithm

- Solver uses the Generalized Reduced Gradient (GRG) algorithm to solve NLPs.
- GRG can also be used on LPs but is slower than the Simplex method.
- The following discussion gives a general (but somewhat imprecise) idea of how GRG works.

An NLP Solution Strategy



Local vs. Global Optimal Solutions



Comments About NLP Algorithms

- It is not always best to move in the direction producing the fastest rate of improvement in the objective.
- NLP algorithms can terminate at local optimal solutions.
- The starting point influences the local optimal solution obtained.

Comments About Starting Points

- The null starting point should be avoided.
- When possible, it is best to use starting values of approximately the same magnitude as the expected optimal values.

A Note About "Optimal" Solutions

- When solving a NLP problem, Solver normally stops when the first of three numerical tests is satisfied, causing one of the following three completion messages to appear:

1) **"Solver found a solution. All constraints and optimality conditions are satisfied."**

This means Solver found a local optimal solution, but does not guarantee that the solution is the global optimal solution.

A Note About "Optimal" Solutions

- When solving a NLP problem, Solver normally stops when the first of three numerical tests is satisfied, causing one of the following three completion messages to appear:

2) **"Solver has converged to the current solution. All constraints are satisfied."**

This means the objective function value changed very slowly for the last few iterations.

A Note About "Optimal" Solutions

- When solving a NLP problem, Solver normally stops when the first of three numerical tests is satisfied, causing one of the following three completion messages to appear:

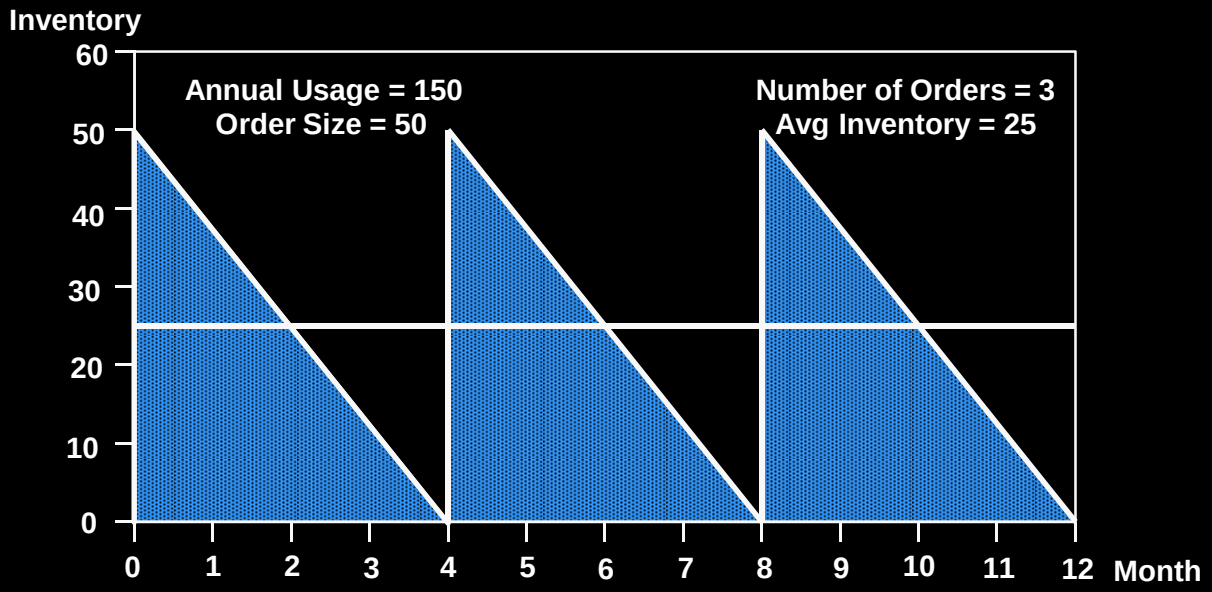
3) "**Solver cannot improve the current solution. All constraints are satisfied**."

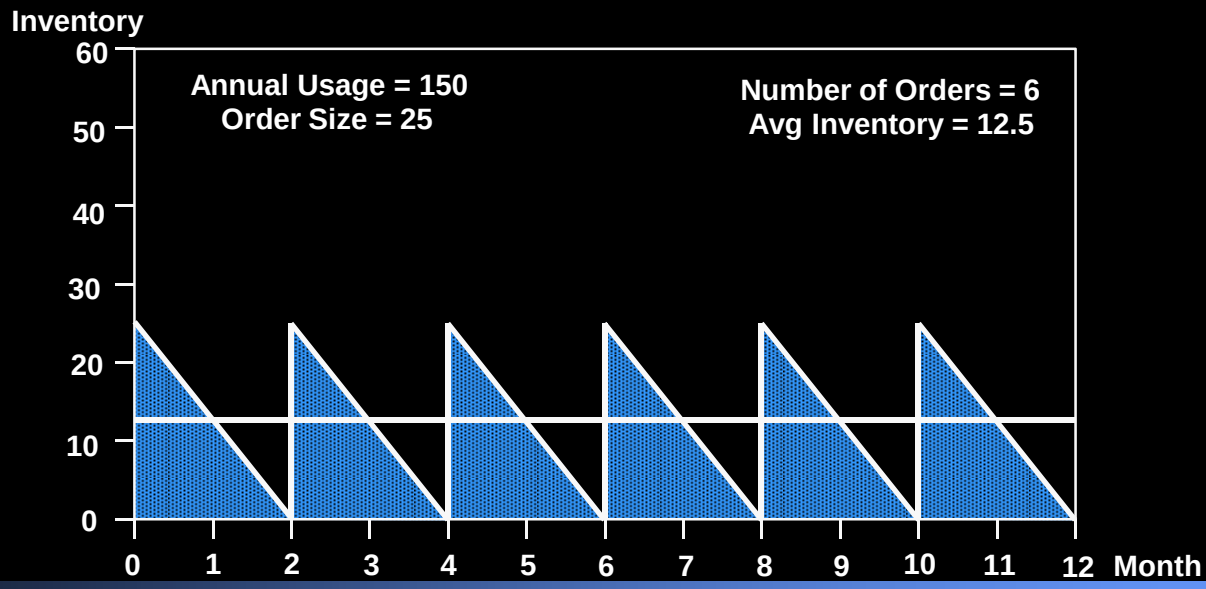
This rare message means the your model is degenerate and the Solver is cycling. Degeneracy can often be eliminated by removing redundant constraints in a model.

The Economic Order Quantity (EOQ) Problem

- Involves determining the optimal quantity to purchase when orders are placed.
- Small orders result in:
  - low inventory levels & carrying costs
  - frequent orders & higher ordering costs
- Large orders result in:
  - higher inventory levels & carrying costs
  - infrequent orders & lower ordering costs

Sample Inventory Profiles





The EOQ Model

where:

D = **annual demand for the item**

C = **unit purchase cost for the item**

S = **fixed cost of placing an order**

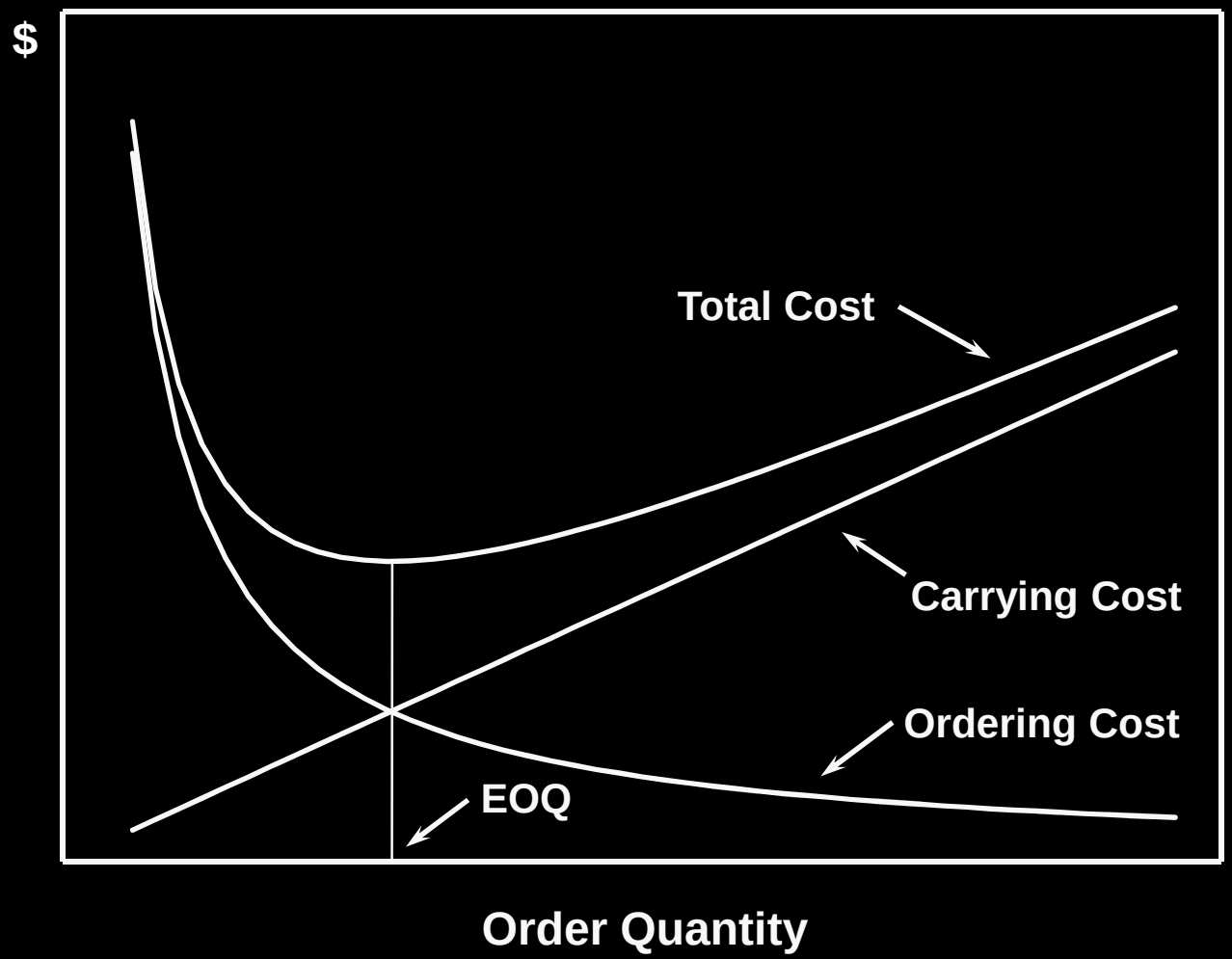
i = **cost of holding inventory for a year (expressed as a % of C)**

Q = **order quantity**

Assumes:

- **Demand (or use) is constant over the year.**
- **New orders are received in full when the inventory level drops to zero.**

EOQ Cost Relationships



An EOQ Example: Ordering Paper For MetroBank

- Alan Wang purchases paper for copy machines and laser printers at MetroBank.
  - Annual demand (D) is for 24,000 boxes
  - Each box costs $35 (C)
  - Each order costs $50 (S)
  - Inventory carrying costs are 18% (i)
- What is the optimal order quantity (Q)?

The Model

(Note the nonlinear objective!)

Implementing the Model

See file Fig8-6.xls

Comments on the EOQ Model

- Using calculus, it can be shown that the optimal value of Q is:

- Numerous variations on the basic EOQ model exist accounting for:
  - quantity discounts
  - storage restrictions
  - backlogging
  - etc

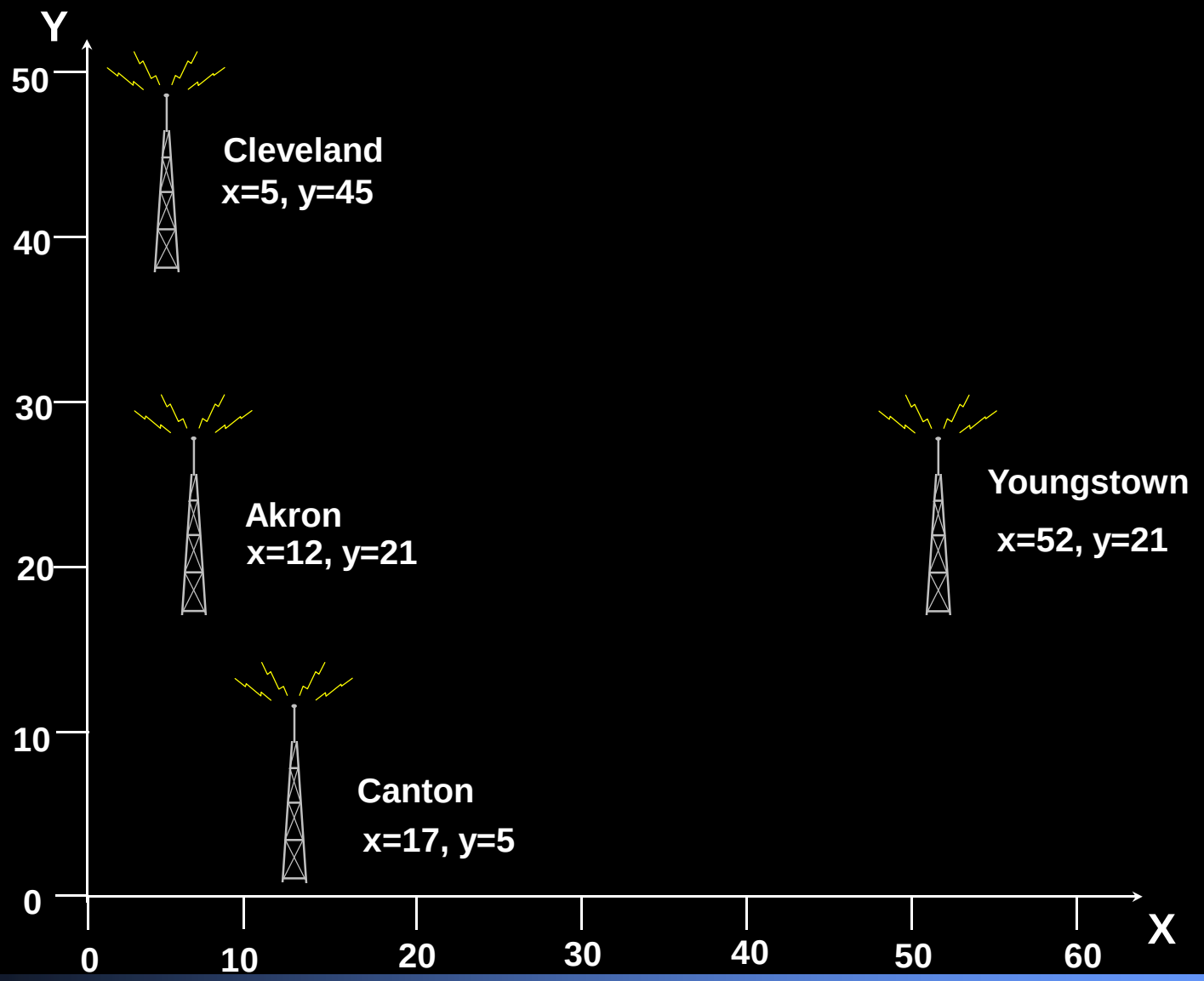
Location Problems

- Many decision problems involve determining optimal locations for facilities or service centers. For example,
  - Manufacturing plants
  - Warehouse
  - Fire stations
  - Ambulance centers
- These problems usually involve distance measures in the objective and/or constraints.
- The straight line (Euclidean) distance between two points (X_1, Y_1) and (X_2, Y_2) is:

A Location Problem: Rappaport Communications

- Rappaport Communications provides cellular phone service in several mid-western states.
- The want to expand to provide inter-city service between four cities in northern Ohio.
- A new communications tower must be built to handle these inter-city calls.
- The tower will have a 40 mile transmission radius.

Graph of the Tower Location Problem



Defining the Decision Variables

X_1 = location of the new tower with respect to the X-axis

Y_1 = location of the new tower with respect to the Y-axis

Defining the Objective Function

- Minimize the total distance from the new tower to the existing towers

MIN:

Defining the Constraints

- Cleveland
- Akron
- Canton
- Youngstown

Implementing the Model

See file Fig8-10.xls

Analyzing the Solution

- The optimal location of the "new tower" is in virtually the same location as the existing Akron tower.
- Maybe they should just upgrade the Akron tower.
- The maximum distance is 39.8 miles to Youngstown.
- This is pressing the 40 mile transmission radius.
- Where should we locate the new tower if we want the maximum distance to the existing towers to be minimized?

Implementing the Model

See file Fig8-13.xls

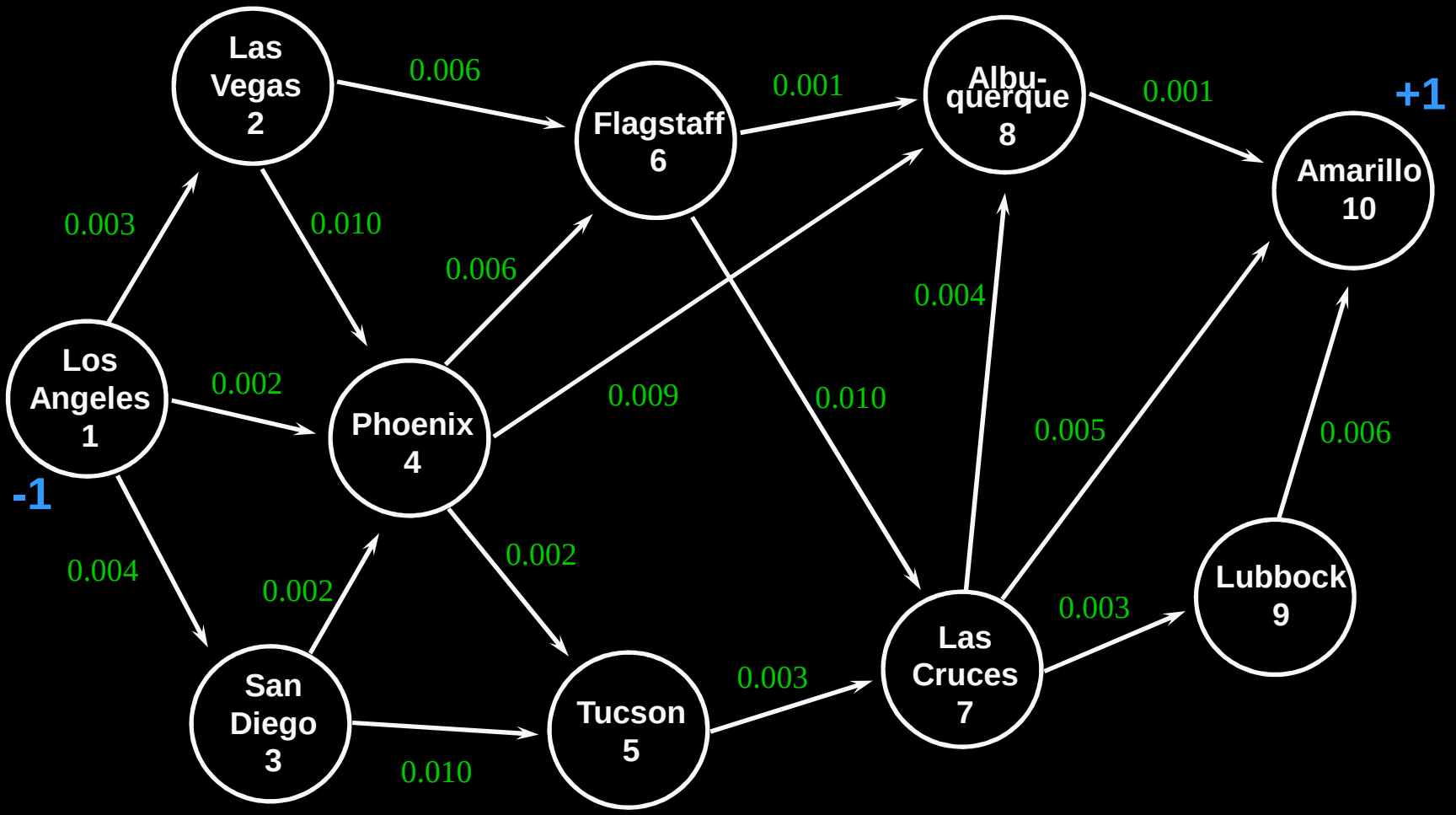
Comments on Location Problems

- The optimal solution to a location problem may not work:
  - The land may not be for sale.
  - The land may not be zoned properly.
  - The "land" may be a lake.
- In such cases, the optimal solution is a good starting point in the search for suitable property.
- Constraints may be added to location problems to eliminate infeasible areas from consideration.

A Nonlinear Network Flow Problem: The SafetyTrans Company

- SafetyTrans specialized in trucking extremely valuable and extremely hazardous materials.
- It is imperative for the company to avoid accidents:
  - It protects their reputation.
  - It keeps insurance premiums down.
  - The potential environmental consequences of an accident are disastrous.
- The company maintains a database of highway accident data which it uses to determine safest routes.
- They currently need to determine the safest route between Los Angeles, CA and Amarillo, TX.

Network for the SafetyTrans Problem



Numbers on arcs represent the probability of an accident occurring.

Defining the Decision Variables

Defining the Objective

Select the safest route by maximizing the probability of not having an accident,

MAX: $(1-P_{12}Y_{12})(1-P_{13}Y_{13})(1-P_{14}Y_{14})(1-P_{24}Y_{24})...(1-P_{9,10}Y_{9,10})$

where:

P_{ij} = probability of having an accident while traveling between node i and node j

Defining the Constraints

- Flow Constraints

$-Y_{12} - Y_{13} - Y_{14} = -1$ } node 1

$+Y_{12} - Y_{24} - Y_{26} = 0$ } node 2

$+Y_{13} - Y_{34} - Y_{35} = 0$ } node 3

$+Y_{14} + Y_{24} + Y_{34} - Y_{45} - Y_{46} - Y_{48} = 0$ } node 4

$+Y_{35} + Y_{45} - Y_{57} = 0$ } node 5

$+Y_{26} + Y_{46} - Y_{67} - Y_{68} = 0$ } node 6

$+Y_{57} + Y_{67} - Y_{78} - Y_{79} - Y_{7,10} = 0$ } node 7

$+Y_{48} + Y_{68} + Y_{78} - Y_{8,10} = 0$ } node 8

$+Y_{79} - Y_{9,10} = 0$ } node 9

$+Y_{7,10} + Y_{8,10} + Y_{9,10} = 1$ } node 10

Implementing the Model

See file Fig8-15.xls

Comments on Nonlinear Network Flow Problems

- Small differences in probabilities can mean large differences in expected values:

  0.9900 * $30,000,000 = $300,000

  0.9626 * $30,000,000 = $1,122,000

- This type of problem is also useful in reliability network problems (e.g., finding the weakest "link" (or path) in a production system or telecommunications network).

A Project Selection Problem: The TMC Corporation

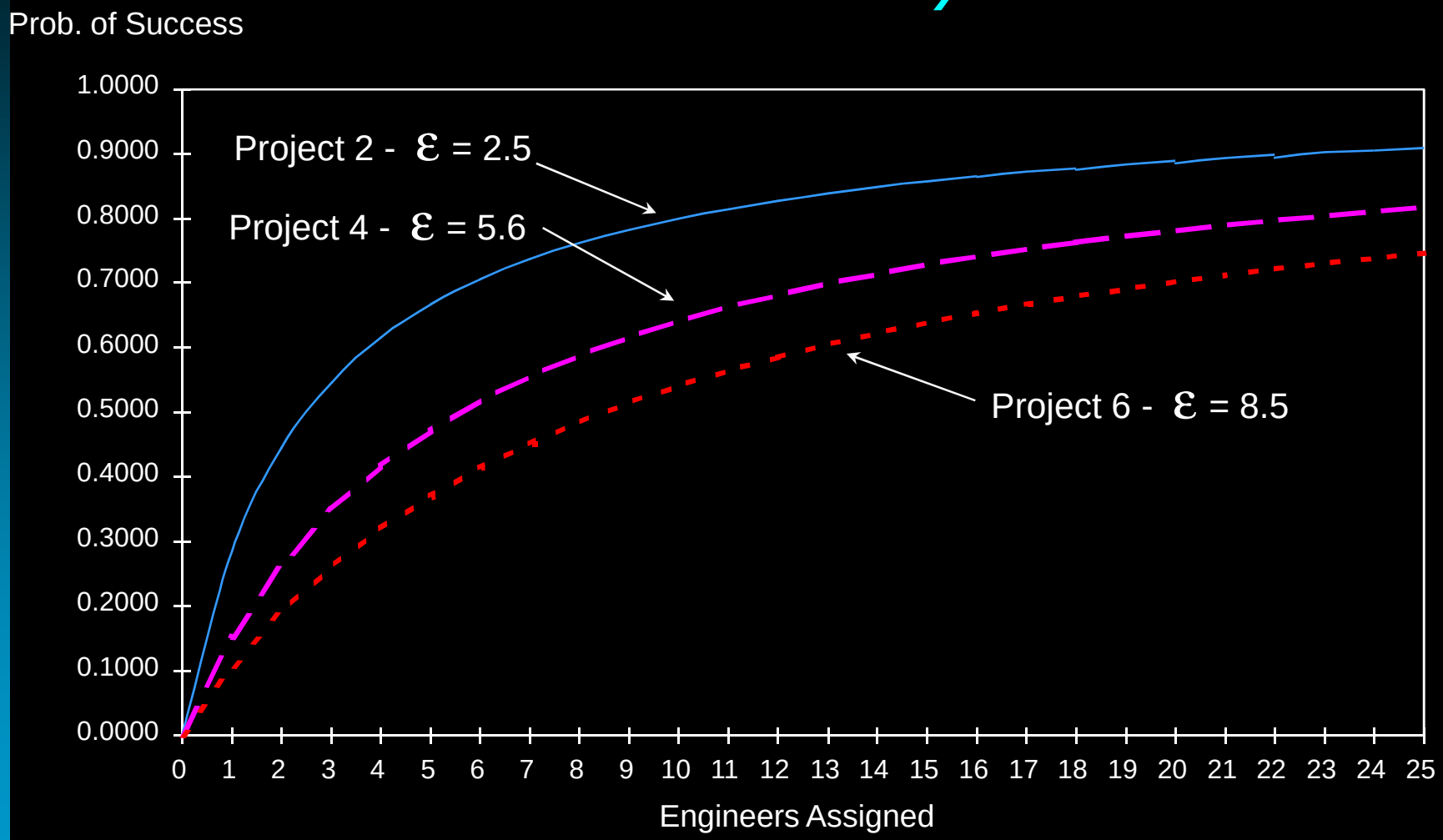
- TMC needs to allocate $1.7 million of R&D budget and up to 25 engineers among 6 projects.
- The probability of success for each project depends on the number of engineers assigned (X_i) and is defined as:

$$P_i = X_i/(X_i + \varepsilon_i)$$

Project	1	2	3	4	5	6
Startup Costs	$325	$200	$490	$125	$710	$240
NPV if successful	$750	$120	$900	$400	$1,110	$800
Probability Parameter ε_i	3.1	2.5	4.5	5.6	8.2	8.5

(all monetary values are in $1,000s)

Selected Probability Functions



Defining the Decision Variables

X_i = the number of engineers assigned to project i, $i = 1, 2, 3, \ldots, 6$

Defining the Objective

Maximize the expected total NPV of selected projects

Defining the Constraints

- Startup Funds

  $325Y_1 + 200Y_2 + 490Y_3 + 125Y_4 + 710Y_5 + 240Y_6 <= 1700$

- Engineers

  $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 <= 25$

- Linking Constraints

  $X_i - 25Y_i <= 0, \quad i = 1, 2, 3, \ldots 6$

- Note: The following constraint could be used in place of the last two constraints...

  $X_1Y_1 + X_2Y_2 + X_3Y_3 + X_4Y_4 + X_5Y_5 + X_6Y_6 <= 25$

  However, this constraint is nonlinear. It is generally better to keep things linear where possible.

Implementing the Model

See file Fig8-19.xls

Optimizing Existing Financial Models

- It is not necessary to always write out the algebraic formulation of an optimization problem, although doing so ensures a thorough understanding of the problem.
- Solver can be used to optimize a host of pre-existing spreadsheet models which are inherently nonlinear.

A Life Insurance Funding Problem

- Thom Pearman owns a whole life policy with surrender value of $6,000 and death benefit of $40,000.
- He'd like to cash in his whole life policy and use interest on the surrender value to pay premiums on a a term life policy with a death benefit of $350,000.
- The premiums on the new policy for the next 10 years are:

Year	1	2	3	4	5	6	7	8	9	10
Premium	$423	$457	$489	$516	$530	$558	$595	$618	$660	$716

- Thom's marginal tax rate is 28%.
- What rate of return will be required on his $6,000 investment?

Implementing the Model

See file Fig8-22.xls

The Portfolio Optimization Problem

- A financial planner wants to create the least risky portfolio with at least a 12% expected return using the following stocks.

	Annual Return		
Year	IBC	NMC	NBS
1	11.2%	8.0%	10.9%
2	10.8%	9.2%	22.0%
3	11.6%	6.6%	37.9%
4	-1.6%	18.5%	-11.8%
5	-4.1%	7.4%	12.9%
6	8.6%	13.0%	-7.5%
7	6.8%	22.0%	9.3%
8	11.9%	14.0%	48.7%
9	12.0%	20.5%	-1.9%
10	8.3%	14.0%	19.1%
11	6.0%	19.0%	-3.4%
12	10.2%	9.0%	43.0%
Avg	7.64%	13.43%	14.93%

	Covariance Matrix		
	IBC	NMC	NBS
IBC	0.00258	-0.00025	0.00440
NMC	-0.00025	0.00276	-0.00542
NBS	0.00440	-0.00542	0.03677

Defining the Decision Variables

p_1 = proportion of funds invested in IBC

p_2 = proportion of funds invested in NMC

p_3 = proportion of funds invested in NBS

Defining the Objective

Minimize the portfolio variance (risk).

Defining the Constraints

- Expected return

  $0.0764\,p_1 + 0.1343\,p_2 + 0.1493\,p_3 >= 0.12$

- Proportions

  $p_1 + p_2 + p_3 = 1$

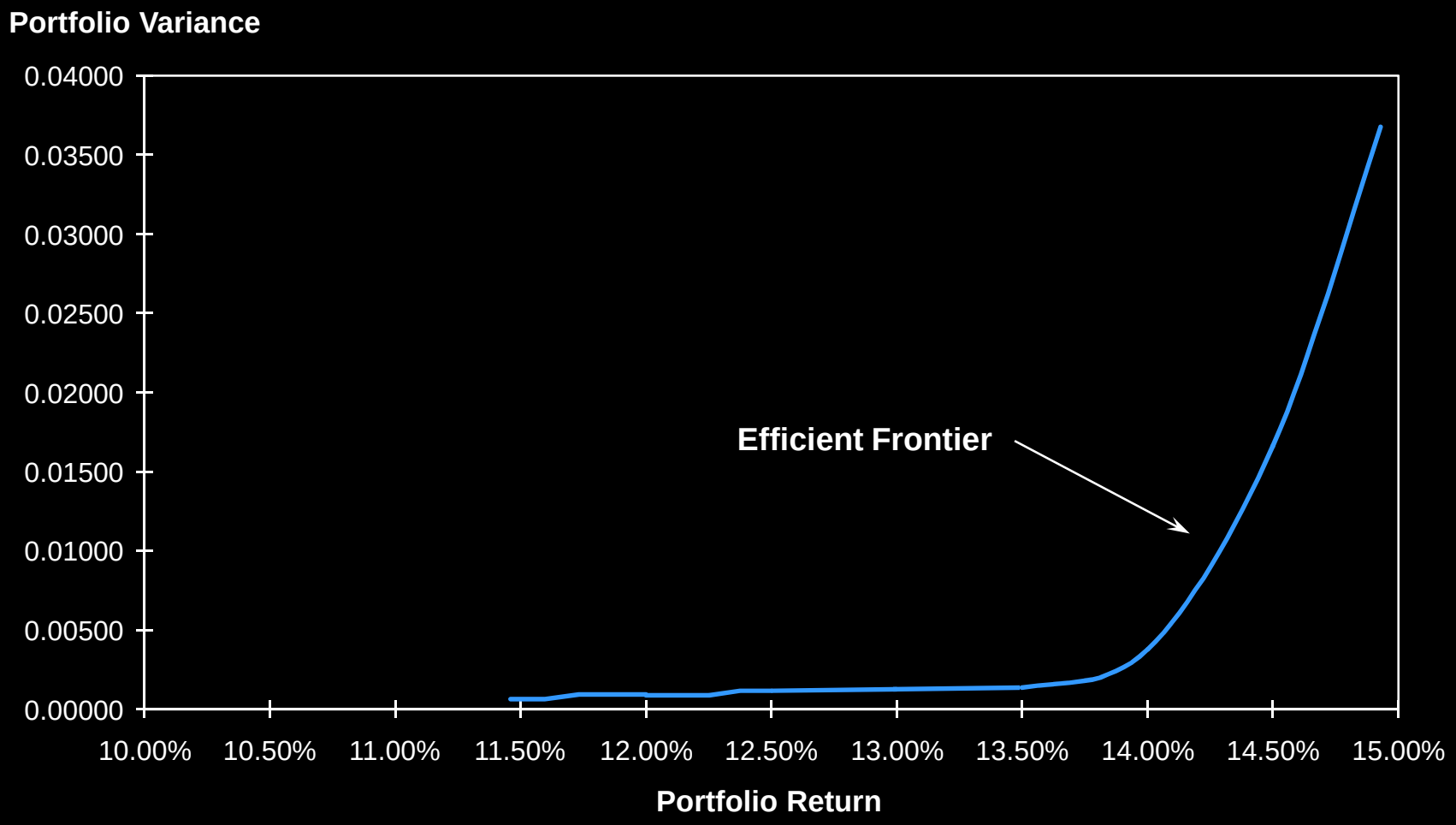
  $p_1, p_2, p_3 >= 0$

  $p_1, p_2, p_3 <= 1$

Implementing the Model

See file Fig8-26.xls

The Efficient Frontier



Multiple Objectives in Portfolio Optimization

- In portfolio problems we usually want to either:
  - Minimize risk (portfolio variance)
  - Maximize the expected return
- We can deal with both objectives simultaneously as follows to generate efficient solutions:

MAX: (1-r)(Expected Return) - r(Portfolio Variance)

S.T.: $p_1 + p_2 + \ldots + p_m = 1$

$p_i >= 0$

where:

0<= r <=1 is a user defined risk aversion value

Note: If r = 1 we minimize the portfolio variance.

If r = 0 we maximize the expected return.

Implementing the Model

See file Fig8-30.xls

Sensitivity Analysis

LP Term	NLP Term	Meaning
Shadow Price	**Lagrange Multiplier**	**Marginal value of resources.**
Reduced Cost	**Reduced Gradient**	**Impact on objective of small changes in optimal values of decision variables.**

- Less sensitivity analysis information is available with NLPs vs. LPs.
- See file Fig8-32.xls

Evolutionary Algorithms

- A technique of heuristic mathematical optimization based on Darwin's Theory of Evolution.
- Can be used on any spreadsheet model, including those with "If" and/or "Lookup" functions.
- Also known as Genetic Algorithms (GAs).

Evolutionary Algorithms

- Solutions to a MP problem can be represented as a vector of numbers (like a chromosome)
- Each chromosome has an associated "fitness" (obj) value
- GAs start with a random population of chromosomes & apply
  - Crossover - exchange of values between solution vectors
  - Mutation - random replacement of values in a solution vector
- The most fit chromosomes survive to the next generation, and the process is repeated

INITIAL POPULATION

Chromosome	X_1	X_2	X_3	X_4	Fitness
1	7.84	24.39	28.95	6.62	282.08
2	10.26	16.36	31.26	3.55	293.38
3	3.88	23.03	25.92	6.76	223.31
4	9.51	19.51	26.23	2.64	331.28
5	5.96	19.52	33.83	6.89	453.57
6	4.77	18.31	26.21	5.59	229.49

CROSSOVER & MUTATION

Chromosome	X_1	X_2	X_3	X_4	Fitness
1	7.84	24.39	31.26	3.55	334.28
2	10.26	16.36	28.95	6.62	227.04
3	3.88	**19.75**	25.92	6.76	301.44
4	9.51	19.51	**32.23**	2.64	495.52
5	4.77	18.31	33.83	6.89	332.38
6	5.96	19.52	26.21	4.60	444.21

Crossover

Mutation

NEW POPULATION

Chromosome	X_1	X_2	X_3	X_4	Fitness
1	7.84	24.39	31.26	3.55	334.28
2	10.26	16.36	31.26	3.55	293.38
3	3.88	**19.75**	25.92	6.76	301.44
4	9.51	19.51	**32.23**	2.64	495.52
5	5.96	19.52	33.83	6.89	453.57
6	5.96	19.52	26.21	4.60	444.21

Example: Beating The Market

- An investor would like to determine portfolio allocations that maximizes the number of times his portfolio outperforms the S&T 500.

See file Fig8-37.xls

The Traveling Salesperson Problem

- A salesperson wants to find the least costly route for visiting clients in n different cities, visiting each city exactly once before returning home.

n	*(n-1)!*
3	*2*
5	*24*
9	*40,320*
13	*479,001,600*
17	*20,922,789,888,000*
20	*121,645,100,408,832,000*

Example: The Traveling Salesperson Problem

- Wolverine Manufacturing needs to determine the shortest tour for a drill bit to drill 9 holes in a fiberglass panel.

See file Fig8-40.xls

End of Chapter 8